# On Learning to Coordinate:
## Random Bits, Normal Forms, & Isomorphisms

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# Coordination (Montagna & Osherson, 1999)

- Informally: Two players or, synonymously, coordinators simultaneously play bits (elements of $\{0, 1\}$) each taking into account the bits played by the other.

- Goal: successful coordination is to (learn to) have their bit streams perfectly match past some point.

- E.g., 2 people show up in the park each day at one of noon (bit 0) or 6pm (bit 1); each silently watches the others past behavior (showing up or not); and each tries, based on the past behavior of the other, to show up exactly when the other shows up.

- Players are modeled as (partial) functions from (finite) bit strings to bits (or bit strings). A player can be, for example, total, partial computable, computable, or probabilistic (the latter with some probability of success).

# A Few Random Bits Help

Our focus is on coordinators which do or do not (learn to) coordinate with a whole class $\mathcal{C}$ of coordinators. E.g., some algorithmic player coordinates with $\mathcal{C}$ = class of polytime comp. 0-1 valued funs.

Surprising result below shows computable probabilistic coordinators beat computable deterministic ones — and with few random bits.

Theorem: Suppose $0 \leq p < 1$.

There exists a class $\mathcal{C}$ of computable deterministic players such that

1. No computable deterministic player can coordinate with all of $\mathcal{C}$.

2. There exists a probabilistic player $PM$ which, can coordinate with each $P$ in $\mathcal{C}$, with probability $\geq p$.

Furthermore, $PM$ above is

– blind (i.e., outputs depends only on input length)

– $k$-memory limited (i.e., does not need to remember too far back)

– uses only $k$ random bits

(for $1 - 2^{-k} \geq p$).

# Stronger Quantifier Order

The previous Theorem featured the quantifier order: $(\forall \text{ probabilities } p < 1)(\exists \mathcal{C})[\cdots]$. The next one improves this to $(\exists \mathcal{C})(\forall \text{ probabilities } p < 1)[\cdots]$ — at some apparent costs re blindness and memory limitation. Open: Are these costs necessary?

Again, but with the stronger quantifier order, computable probabilistic coordinators beat computable deterministic ones — and with just a few random bits.

Theorem: There exists a class $\mathcal{C}$ of computable deterministic players such that

1. No computable deterministic player can coordinate with all of $\mathcal{C}$.

2. For all $p$ such that $0 \leq p < 1$, there exists a probabilistic player $PM$ which, can coordinate with each $P$ in $\mathcal{C}$, with probability $\geq p$.

Furthermore, $PM$ above is

– $k$-memory limited only after the first 0 it sees

– uses only $k$ random bits

(for $1 - 2^{-k} \geq p$).

# Teams of Coordinators

For $1 \leq \ell \leq m$, $\mathbf{Team}^{\ell}_{m}\mathbf{Coord}$ denotes the collection of classes $\mathcal{C}$ of algorithmic *deterministic* players such that we have $m$ deterministic algorithmic coordinators $M^1, \ldots, M^m$ so that for each element $F$ of $\mathcal{C}$, at least $\ell$ of $M^1, \ldots, M^m$ coordinate with $F$.

Motivation: In the more typical inductive inference setting of learning programs for total computable functions, team learning by computable deterministic machines is completely characterized by single machines learning probabilistically and vice versa (Pitt, 1984, 1989; Pitt and Smith 1988).

The analog of $\mathbf{Team}^{\ell}_{m}$ in that context has exactly the inferring power of single machines which succeed with probability at least $\frac{\ell}{m}$.

One might suspect that such a characterization holds for learning to coordinate, and that the theorems of the last two slides are readily explainable as deterministic team learning in disguise.

We will see on the next slide that no such characterization holds in the present setting of learning to coordinate!

Theorem: Suppose $0 \leq p < 1$.

There exists a class $\mathcal{C}$ of computable deterministic players such that

1. $\mathcal{C} \notin \mathbf{Team}^1_m\mathbf{Coord}$.

2. $\mathcal{C} \in \mathbf{Team}^1_{m+1}\mathbf{Coord}$.

3. There exists a probabilistic player $PM$ which, can coordinate with $P$ in $\mathcal{C}$ with probability $\geq p$.

Furthermore, $PM$ above is

– blind

– $k$-memory limited

– uses only $k$ random bits

(for $\frac{2^k - m}{2^k} \geq p$).

The team in clause (2) above also is blind, and memory limited (uses $\lceil \log_2(m+1) \rceil$ bits).

ALSO: a stronger quantifier order variant holds of above theorem — see conference paper.

# Comparisons

$\mathbf{Prob}_p\mathbf{Coord}$ denotes the collection of classes $\mathcal{C}$ of computable *deterministic* players such that some algorithmic probabilistic coordinator coordinates with each element of $\mathcal{C}$ with probability $\geq p$.

Theorem: Suppose $0 \leq p < q \leq 1$.

Then, there is a class of computable deterministic players $\in (\mathbf{Prob}_p\mathbf{Coord} - \mathbf{Prob}_q\mathbf{Coord})$.

Moreover, for $\ell, m$ such that $p \leq \frac{\ell}{m} < q$ and $k$ such that $k = \lceil \log_2(m) \rceil$, the positive half of this corollary is witnessed by a blind probabilistic coordinator which employs only $k$ random bits and is $k$-memory limited.

Theorem: Suppose $\ell, m, v, w$ are positive integers, $\ell \leq m$, $v \leq w$.

$\mathbf{Team}_m^\ell\mathbf{Coord} \subseteq \mathbf{Team}_w^v\mathbf{Coord}$ iff there exists a way to distribute $w$ balls among $m$ boxes such that any combination of $l$ boxes receives at least $v$ balls.

# Normal Form for Total Players

Notation: Consider indexed family $\mathcal{C}$ whose members are $\lambda x.C(i,x)$.

Let $LE(\mathcal{C})$ be the player which NV-identifies $\mathcal{C}$ using identification by enumeration strategy.

Theorem:

For every total computable player $P$, there exists a dense indexed family $\mathcal{C}_P$ and corresponding enumerating function $\lambda x.C_P(i,x)$ such that

$P = LE(\mathcal{C}_P)$.

Warning: Complexity of the normal form may be exponential.

# Capability of Total Players

Notation:

$\mathbf{SCOPE}(P) = \{Q : Q$ is a total player with which $P$ coordinates $\}$.

Theorem: Suppose $P$ and $Q$ are arbitrary total players.

Then $\mathbf{SCOPE}(P)$ and $\mathbf{SCOPE}(Q)$ are $P,Q$-computably homeomorphic. In particular if $P,Q$ are computable then $\mathbf{SCOPE}(P)$ and $\mathbf{SCOPE}(Q)$ are computably homeomorphic.

Notation:

$IND(P) = \{i : \varphi_i$ is a total player with which $P$ coordinates $\}$.

Theorem: Suppose $P$ is a computable total player.

Then $IND(P)$ is 2-$\Sigma^0_2$-complete.